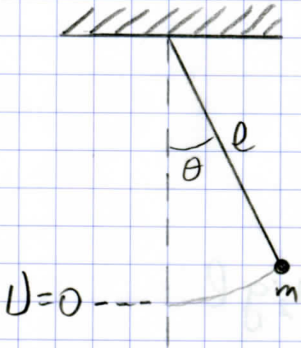
TM5 Pr 7.26

DETERMINE THE HAMILTONIAN AND HAMILTON'S EQUATIONS OF MOTION FOR
a) A SIMPLE PENDULUM AND
b) A SIMPLE ATWOOD MACHINE



a) THE SIMPLE PENDULUM

$$T = \frac{1}{2} m \dot{s}^2 = \frac{1}{2} m \ell^2 \dot{\theta}^2$$

$$U = mg\ell(1-\cos\theta)$$

THE LAGRANGIAN IS

$$L = \frac{1}{2} m\ell^2\dot{\theta}^2 - mg\ell(1-\cos\theta)$$

FIND THE MOMENTUM

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{p_\theta}{m\ell^2}$$

RE-WRITE T IN TERMS OF p_θ

$$T = \frac{1}{2} m\ell^2 \left(\frac{p_\theta}{m\ell^2}\right)^2 = \frac{1}{2}\frac{p^2}{m\ell^2}$$

SINCE $T = T(\dot{\theta}^2)$ AND $U \neq U(\dot{\theta})$

$$H = T + U = \boxed{\frac{1}{2}\frac{p^2}{m\ell^2} + mg\ell(1-\cos\theta) = H}$$

APPLY HAMILTON'S EQUATIONS

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2} \quad \Rightarrow \quad p_\theta = m\ell^2\dot{\theta}$$ WHAT WE HAD EARLIER

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mg\ell\sin\theta$$

PUTTING THESE TOGETHER:

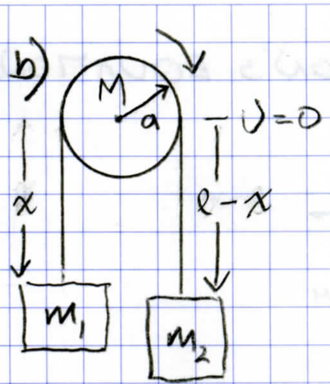
$$\dot{p}_\theta = -mg\ell\sin\theta = m\ell^2\ddot{\theta}$$

$$\Rightarrow \quad \boxed{\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0}$$

A PENDULUM WITH $\omega_N = \sqrt{\frac{g}{\ell}}$!

→

b)



Write energies

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}I\overbrace{\dot{\theta}^2}^{\frac{\dot{x}^2}{a^2}}$$

$$= \frac{1}{2}\left(m_1 + m_2 + \frac{I}{a^2}\right)\dot{x}^2$$

$$U = -m_1 g x - m_2 g(l - x)$$

The Lagrangian is

$$L = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{a^2}\right)\dot{x}^2 + g(m_1 - m_2)x + m_2 g l$$

Find p_x

$$p_x = \frac{\partial L}{\partial \dot{x}} = \left(m_1 + m_2 + \frac{I}{a^2}\right)\dot{x}$$

$$\Rightarrow \quad \dot{x} = \frac{p_x}{\left(m_1 + m_2 + \frac{I}{a^2}\right)}$$

Since $T = T(\dot{x}^2)$ and $U \neq U(\dot{x})$

$$H = T + U = \boxed{\frac{1}{2}\frac{p_x^2}{m_1 + m_2 + \frac{I}{a^2}} - g(m_1 - m_2)x - m_2 g l = H}$$

Apply Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m_1 + m_2 + \frac{I}{a^2}} \quad \Rightarrow \quad p_x = \left(m_1 + m_2 + \frac{I}{a^2}\right)\dot{x}$$

Yeah, we know that!

$$\dot{p} = -\frac{\partial H}{\partial x} = +g(m_1 - m_2)$$

Putting these together:

$$\dot{p}_x = g(m_1 - m_2) = \left(m_1 + m_2 + \frac{I}{a^2}\right)\ddot{x}$$

$$\Rightarrow \boxed{\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2 + \frac{I}{a}}\, g}$$

The result we got from NSL!